Função Gama

$$\boxed{\Gamma(x+1) = \int_0^\infty t^x e^{-t}\, dt}$$

$$\boxed{\Gamma(n+1) = n!}$$

$(x=0)$ $\quad \Gamma(1) = \int_0^\infty e^{-t}\, dt = e^{-t}\Big|_1^0 = 1$

$(x=1)$ $\quad \Gamma(2) = \int_0^\infty t\, \underbrace{e^{-t} dt}_{dv \Rightarrow v = -e^{-t}} = \int_0^\infty e^{-t} dt = \Gamma(1) = 1$

$(x=2)$ $\quad \Gamma(3) = \int_0^\infty \underset{u}{t^2}\, e^{-t}\, dt = 2 \underbrace{\int_0^\infty t\, e^{-t}\, dt}_{\Gamma(2)} = 2\,\Gamma(2) = 2$

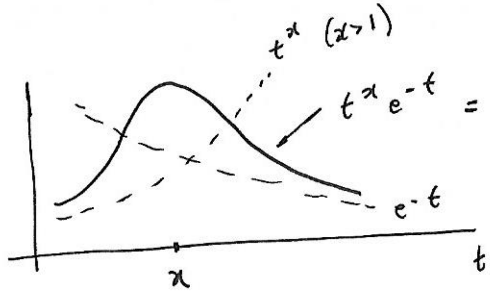
$(x=3)$ $\quad \Gamma(4) = \int_0^\infty t^3 e^{-t}\, dt = 3 \underbrace{\int_0^\infty t^2 e^{-t}\, dt}_{\Gamma(3)} = 3\,\Gamma(3) = 3 \times 2\,\Gamma(2) = 3!$

$(x=n-1)$ $\quad \Gamma(n) = (n-1)\,\Gamma(n-1) = (n-1)!$

$(x=n)$ $\quad \Gamma(n+1) = \int_0^\infty t^n e^{-t}\, dt = n \underbrace{\int_0^\infty t^{n-1} e^{-t}\, dt}_{\Gamma(n)} = n\,\Gamma(n) = n\,(n-1)! = n!$

Fórmula de Stirling

$$\Gamma(x+1) = \int_0^{\infty} t^x e^{-t}\, dt$$



$$t^x e^{-t} = e^{\underbrace{(x \ln t - t)}_{f(x,t)}}$$

Expandindo em torno do máximo $(t = t^* = x)$

$$f(x,t) = f(x,t^*) + (t-t^*)\cancel{f'(x,t^*)}^{\,0} + \frac{1}{2}(t-t^*)^2 \underbrace{f''(x,t^*)}_{-1/x}$$

$$= x\ln x - x - \frac{1}{2x}(t-x)^2$$

$$\frac{d}{dt}\left[e^{f(x,t)}\right] = 0 \qquad (x \gg 1)$$

$$e^{f(x,t)}\frac{df}{dt} = e^{f(x,t)}\left(\frac{x}{t} - 1\right) = 0$$

$t = x$ (máximo)

$$\frac{d^2}{dt^2}\left[e^{f(x,t)}\right] = \frac{d}{dt}\frac{d}{dt}\left[e^{f(x,t)}\right]$$

$$= \frac{d}{dt}\left[e^{f(x,t)}\frac{df}{dt}\right]$$

$$= \frac{d}{dt}\left[e^{f(x,t)}\right]\frac{df}{dt} + e^{f(x,t)}\frac{d^2 f}{dt^2}$$

$$= e^{f(x,t)}\left\{\cancel{\left(\frac{df}{dt}\right)^2}^{\,0} + \frac{d^2 f}{dt^2}\right\}\Bigg|_{t=x}$$

$$= e^{f(x,t)}\left(\frac{-x}{t^2}\right)\Bigg|_{t=x} < 0 \quad \text{(máximo)}$$

$$\Gamma(x+1) = \int_0^{\infty} e^{\left[x\ln x - x - \frac{1}{2x}(t-x)^2\right]} dt$$

$$= x^x e^{-x} \underbrace{\int_0^{\infty} e^{-\frac{1}{2x}(t-x)^2} dt}_{\sqrt{2\pi x}}$$

$$\boxed{\Gamma(x+1) = \sqrt{2\pi x}\; x^x e^{-x}}$$ (Fórmula de Stirling) $(x \gg 1)$

$$\ln \Gamma(x+1) = \underbrace{\frac{1}{2}\ln x + x\ln x}_{\left(\frac{1}{2}+x\right)\ln x \,\simeq\, x\ln x} - x = \boxed{x\ln x - x = x(\ln x - 1)}$$

Volume da esfera em D dimensões

$$\begin{cases} d^3\vec{p} = \Omega_3\, p^2\, dp \quad (D=3) \quad \Longrightarrow \quad d^D\vec{p} = \Omega_D\, p^{D-1}\, dp \quad (\text{D dimensões}) \\[2ex] V_D = \Omega_D \displaystyle\int_0^P p^{D-1}\, dp = \Omega_D \frac{p^D}{D}\Big|_0^P = \Omega_D \frac{P^D}{D} \end{cases}$$

raio da esfera

$$p^2 = \sum_{i=1}^{D} p_i^2 \Longrightarrow \int e^{-p^2} \underbrace{d^D\vec{p}}_{dp_1 dp_2 \cdots dp_D} = \Big[\underbrace{\int_{-\infty}^{\infty} e^{-\xi^2} d\xi}_{\sqrt{\pi}}\Big]^D = \pi^{D/2}$$

$$\Omega_D \int_0^\infty e^{-p^2} p^{D-1} dp = \Omega_D \int_0^\infty e^{-p^2} (p^2)^{D/2} p^{-1} \underbrace{p^{-1} p}_{1} \overbrace{dp}^{\frac{1}{2} d(p^2)} = \frac{\Omega_D}{2} \int_0^\infty e^{-p^2} (p^2)^{D/2-1} d(p^2)$$

$$= \frac{\Omega_D}{2} \underbrace{\int_0^\infty e^{-t}\, t^{D/2-1}\, dt}_{\Gamma(D/2)} = \pi^{D/2}$$

$$\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)} \Longrightarrow V_D = \frac{\pi^{D/2} P^D}{D/2\; \Gamma(D/2)} = \boxed{\frac{\pi^{D/2} P^D}{\Gamma(D/2+1)} = V_D}$$

Função Zeta de Riemann

$$\boxed{\zeta(n) = \sum_{l=1}^{\infty} \frac{1}{l^n} = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \text{------} \quad (n>1)}$$

$$\zeta(2) = \frac{\pi^2}{6} = 1.645$$

$$\zeta(3) = 1.202$$

$$\zeta(4) = \frac{\pi^4}{90} = 1.082$$

$$\zeta(5) = 1.037$$

$$\zeta(6) = \frac{\pi^6}{941} = 1.011$$

$$\zeta(7) = 1.008$$

$\zeta(3/2) = 2.612$ (maior valor)

$$\zeta(5/2) = 1.341$$

$$\zeta(7/2) = 1.127$$

$$\boxed{\zeta(n) = \frac{1}{\Gamma(n)} \int_0^{\infty} \frac{x^{n-1}}{e^x - 1}\, dx}$$

$$I(\lambda, 1/2) = \int_0^\infty \frac{x^{1/2}}{\lambda^{-1} e^x - 1}\, dx = \lambda \int_0^\infty \frac{x^{1/2}}{e^x - \lambda}\, dx = \Gamma(3/2)\left(\lambda + \frac{\lambda^2}{2^{3/2}} + \frac{\lambda^3}{3^{3/2}} + \cdots\right)$$

$$I(1, 1/2) = \Gamma(3/2)\, \zeta(3/2)$$

$$f(\lambda) = \frac{1}{e^x - \lambda} = (e^x - \lambda)^{-1} \implies f(0) = e^{-x} \qquad f'(\lambda) = (e^x - \lambda)^{-2} \implies f'(0) = e^{-2x}$$

$$f''(\lambda) = 2(e^x - \lambda)^{-3} \implies f''(0) = 2e^{-3x} \qquad f'''(\lambda) = 3 \times 2\, (e^x - \lambda)^{-4} \implies f'''(0) = 3!\, e^{-4x}$$

$$f(\lambda) = e^{-x} + \lambda e^{-2x} + \lambda^2 e^{-3x} + \lambda^3 e^{-4x} + \cdots$$

$$\lambda \int_0^\infty \frac{x^{1/2}}{e^x - \lambda}\, dx = \lambda \underbrace{\int_0^\infty x^{1/2} e^{-x}\, dx}_{\Gamma(3/2)} + \lambda^2 \underbrace{\int_0^\infty x^{1/2} e^{-\overset{y}{\widetilde{2x}}}\, dx}_{\frac{1}{2^{3/2}}\Gamma(3/2)} + \lambda^3 \underbrace{\int_0^\infty x^{1/2} e^{-3x}\, dx}_{\frac{1}{3^{3/2}}\Gamma(3/2)} + \cdots$$

$$\boxed{\lambda \int_0^\infty \frac{x^{1/2}}{e^x - \lambda}\, dx = \Gamma(3/2)\left(\lambda + \frac{\lambda^2}{2^{3/2}} + \frac{\lambda^3}{3^{3/2}} + \cdots\right)}$$

$$\lambda \int_0^\infty \frac{x^{3/2}}{e^x - \lambda}\, dx = \lambda \underbrace{\int_0^\infty x^{3/2} e^{-x}\, dx}_{\Gamma(5/2)} + \lambda^2 \underbrace{\int_0^\infty x^{3/2} e^{-2x}\, dx}_{1/2^{5/2}\ \Gamma(5/2)} + \lambda^3 \underbrace{\int_0^\infty x^{3/2} e^{-3x}\, dx}_{\frac{1}{3^{5/2}}\Gamma(5/2)} + \cdots$$

$$\boxed{\lambda \int_0^\infty \frac{x^{3/2}}{e^x - \lambda}\, dx = \Gamma(5/2)\left(\lambda + \frac{\lambda^2}{2^{5/2}} + \frac{\lambda^3}{3^{5/2}} + \cdots\right)}$$