Oscilador Harmônico

Método algébrico

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega_0^2 x^2 \implies \frac{H}{\hbar\omega_0} = \mathcal{H} = \left(\frac{1}{2m\hbar\omega_0} p^2 + \frac{m\omega_0}{2\hbar} x^2 \right)$$

$$a_+ = \frac{-i}{\sqrt{2m\hbar\omega_0}} p + \sqrt{\frac{m\omega_0}{2\hbar}}\, x \implies a_+^\dagger = a_- = \frac{i}{\sqrt{2m\hbar\omega_0}} p + \sqrt{\frac{m\omega_0}{2\hbar}}\, x \quad \text{(não hermitianos)}$$

$$\begin{cases} a_+ a_- = \underbrace{\frac{p^2}{2m\hbar\omega_0} + \frac{1}{2}\frac{m\omega_0}{\hbar} x^2}_{\mathcal{H}} + \frac{i}{2\hbar} \underbrace{(xp - px)}_{i\hbar} = \mathcal{H} - \frac{1}{2} \\ \\ a_- a_+ = \mathcal{H} + \frac{i}{2\hbar} \underbrace{(px - xp)}_{-i\hbar} = \mathcal{H} + \frac{1}{2} \end{cases} \quad \text{(hermitianos)}$$

$$\begin{cases} x = \sqrt{\frac{\hbar}{2m\omega_0}} (a_+ + a_-) \\ p = -i \sqrt{\frac{m\hbar\omega_0}{2}} (a_+ - a_-) \end{cases}$$

$$\begin{cases} a_+ a_- = \mathcal{H} - \dfrac{1}{2} \\[2mm] a_- a_+ = \mathcal{H} + \dfrac{1}{2} \end{cases} \Longrightarrow \mathcal{H} = \underbrace{a_+ a_-}_{N} + \frac{1}{2}$$

- $[\mathcal{H}, a_\pm] = [N, a_\pm] = \pm a_\pm$

$$[a_+ a_-, a_\pm] = a_+ a_- a_\pm - a_\pm a_+ a_- + - a_+ a_\pm a_- + a_+ a_\pm a_- = a_+ [a_-, a_\pm] + [a_+, a_\pm] a_-$$

$$\Downarrow$$

$$\begin{cases} [a_+ a_-, a_+] = a_+ [a_-, a_+] = a_+ (\underbrace{a_- a_+ - a_+ a_-}_{1}) \\ [a_+ a_-, a_-] = \underbrace{[a_+, a_-]}_{-1} a_- = -a_- \end{cases}$$

- $H\psi = E\psi \Longrightarrow \mathcal{H}\psi = \varepsilon\psi$ ($\varepsilon > 0$ – energia mínima maior que $V_{min} = 0$)

- $[\mathcal{H}, a_\pm]\psi = \mathcal{H} a_\pm \psi - a_\pm \underbrace{\mathcal{H}\psi}_{\varepsilon\psi} = \pm a_\pm \psi \Longrightarrow \mathcal{H}(a_\pm \psi) = (\varepsilon \pm 1)(a_\pm \psi)$

$$\begin{cases} a_+\psi \text{ – auto estado de } \mathcal{H} \text{ c/ energia } \varepsilon + 1 \\ a_-\psi \text{ – auto estado de } \mathcal{H} \text{ c/ energia } \varepsilon - 1 \end{cases}$$

Determinação do espectro de energia

menor autovalor

$$\left\{\begin{array}{l} \mathcal{H}\psi_0 = \varepsilon_0 \psi_0 \quad \text{estado fundamental} \quad \Longrightarrow \quad a_-\psi_0 = 0 \\ \mathcal{H} = \underbrace{a_+a_-}_{N} + \frac{1}{2} \quad \Longrightarrow \quad \underbrace{\left(N + \tfrac{1}{2}\right)}_{\mathcal{H}}\psi_0 = a_+ \cancel{a_-\psi_0}^{\,0} + \frac{1}{2}\psi_0 = \varepsilon_0 \psi_0 \Rightarrow \boxed{\varepsilon_0 = \tfrac{1}{2}} \Rightarrow \boxed{E_0 = \hbar\frac{\omega_0}{2}} \end{array}\right.$$

$$\underset{\substack{\downarrow \\ N+1/2}}{\mathcal{H}}(a_+\psi_0) = (\varepsilon_0 + 1)(a_+\psi_0) = \left(1 + \tfrac{1}{2}\right)(a_+\psi_0) \Longrightarrow N(a_+\psi_0) = a_+\psi_0 \Longrightarrow N\psi_1 = \psi_1$$

$$\Downarrow$$

$$\mathcal{H}\psi_1 = \underbrace{\left(1 + \tfrac{1}{2}\right)}_{\varepsilon_1 = 1 + \varepsilon_0}\psi_1$$

$$\mathcal{H}(a_+\psi_1) = (\varepsilon_1 + 1)(a_+\psi_1) = \left(2 + \tfrac{1}{2}\right)(a_+\psi_1) \Longrightarrow N(a_+\psi_1) = 2(a_+\psi_1) \Rightarrow N\psi_2 = 2\psi_2$$

$$\Downarrow$$

$$\mathcal{H}\psi_2 = \underbrace{\left(2 + \tfrac{1}{2}\right)}_{\varepsilon_2 = 2 + \varepsilon_0}\psi_2$$

$$\vdots$$

$$\mathcal{H}\psi_n = \underbrace{\left(n + \tfrac{1}{2}\right)}_{\varepsilon_n}\psi_n \Longrightarrow \boxed{E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega_0}$$

Determinação dos autoestados de energia

- Estado fundamental: $a_- \psi_0 = 0 \implies \left(\frac{i}{\sqrt{2m\hbar\omega_0}} \overset{-i\hbar \frac{d}{dx}}{p} + \sqrt{\frac{m\omega_0}{2\hbar}}\, x \right) \psi_0(x) = 0$

$$\frac{d}{dx}\psi_0(x) + \frac{m\omega_0}{\hbar}\psi_0(x) = 0 \implies \frac{1}{\psi_0}\frac{d\psi_0}{dx} = -\frac{m\omega_0}{\hbar}x$$

$$\Downarrow$$

$$\ln \psi_0 = -\frac{m\omega_0}{2\hbar}x^2 + cte$$

$$\boxed{\psi_0(x) = A_0\, e^{-\frac{m\omega_0}{2\hbar}x^2}} \quad (\text{par})$$

normalização: $\int_{-\infty}^{\infty} |\psi_0|^2 dx = A_0^2 \underbrace{\int_{-\infty}^{\infty} e^{-\overset{\alpha}{\frac{m\omega_0}{\hbar}} x^2}}_{\sqrt{\pi/\alpha} = \sqrt{\frac{\pi\hbar}{m\omega_0}}} = 1 \implies A_0 = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4}$

$$\begin{cases} \psi_0(x) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega_0}{2\hbar}x^2} & (\text{par}) \\[2ex] H\psi_0(x) = E_0\psi_0(x) = \frac{\hbar\omega_0}{2}\psi_0(x) \end{cases}$$

Primeiro estado excitado

$$a_+\psi_0 \sim \psi_1 \implies \left(\frac{-i}{\sqrt{2m\hbar\omega_0}}\,p + \sqrt{\frac{m\omega_0}{2\hbar}}\,x\right)\psi_0(x) \sim \psi_1(x) \qquad p = -i\hbar\frac{d}{dx}$$

$$\begin{cases} \psi_0 = A_0\, e^{-\frac{\alpha}{2}x^2} & A_0 = \left(\frac{\alpha}{\pi}\right)^{1/4} \\ \frac{d\psi_0}{dx} = -\alpha x \psi_0 \end{cases}$$

$$\underbrace{\frac{d\psi_0(x)}{dx}}_{-\alpha x\psi_0} - \underbrace{\left(\frac{m\omega_0}{\hbar}\right)}_{\alpha} x\psi_0(x) \sim \psi_1(x)$$

$$\implies \boxed{\psi_1(x) \sim x\,\psi_0(x)} \quad \text{(ímpar)}$$

normalização: $\displaystyle\int_{-\infty}^{\infty} |\psi_1|^2 dx = A_1^2 A_0^2 \underbrace{\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx}_{\sqrt{\frac{\pi}{\alpha}}\frac{1}{2\alpha}} = 1$, $\quad A_0^2 = \sqrt{\frac{\alpha}{\pi}}$

$(\psi_1 = A_1 x \psi_0)$

$$\Downarrow$$

$$\boxed{A_1 = \sqrt{2\alpha}}$$

$$\boxed{\begin{aligned} I(\alpha) &= \int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\pi/\alpha} \\ -\frac{dI}{d\alpha} &= \int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}\frac{1}{2\alpha} \end{aligned}}$$

$$\boxed{\psi_1(x) = \sqrt{2\alpha}\, x\, \psi_0(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} \sqrt{2\alpha}\, x\, e^{-\frac{\alpha}{2}x^2}} \qquad \boxed{\alpha = \frac{m\omega_0}{\hbar}}$$

$$\boxed{H\psi_1(x) = E_1\psi_1(x) = \left(1 + \tfrac{1}{2}\right)\hbar\omega_0\, \psi_1(x)}$$

Propriedades dos operadores a_+ e a_-

- $a_+ \psi_{n-1} \sim \psi_n \iff c_+^n \psi_n = a_+ \psi_{n-1} \qquad (a_- = a_+^{+})$

$$\Downarrow$$

$$\left(c_+^n \psi_n , c_+^n \psi_n\right) = \left(a_+ \psi_{n-1} , a_+ \psi_{n-1}\right) = \left(\psi_{n-1} , \underbrace{a_- a_+}_{\mathcal{H}+1/2} \psi_{n-1}\right)$$

$$|c_+^n|^2 \underbrace{(\psi_n, \psi_n)}_{1} = \left(n - \cancel{1} + \cancel{1/2} + \cancel{1/2}\right) \underbrace{(\psi_{n-1}, \psi_{n-1})}_{1} = \boxed{n = |c_+^n|^2}$$

$$\boxed{a_+ \psi_n = \sqrt{n+1}\ \psi_{n+1}}$$

- $a_- \psi_{n+1} \sim \psi_n \iff c_-^n \psi_n = a_- \psi_{n+1}$

$$\Downarrow$$

$$|c_-^n|^2 \underbrace{(\psi_n, \psi_n)}_{1} = \left(a_- \psi_{n+1} , a_- \psi_{n+1}\right) = \left(\psi_{n+1} , \underbrace{a_+ a_-}_{\mathcal{H}-1/2} \psi_{n+1}\right)$$

$$= \left(n + 1 + \cancel{1/2} - \cancel{1/2}\right) \underbrace{(\psi_{n+1}, \psi_{n+1})}_{1} \implies \boxed{|c_-^n|^2 = n+1}$$

$$\boxed{a_- \psi_n = \sqrt{n}\ \psi_{n-1}}$$

$$\begin{cases} a_+ \psi_n = \sqrt{n+1}\, \psi_{n+1} \\ a_- \psi_n = \sqrt{n}\, \psi_{n-1} \end{cases}$$

- $a_+ \psi_0 = \psi_1$

$$(a_+)^2 \psi_0 = a_+ (a_+ \psi_0) = a_+ \psi_1 = \sqrt{2}\, \psi_2 \implies \psi_2 = \frac{(a_+)^2}{\sqrt{2}} \psi_0 \quad \text{(normalizado)}$$

$$(a_+)^3 \psi_0 = a_+ a_+ (a_+ \psi_0) = a_+ a_+ \psi_1 = \sqrt{2}\, \underbrace{a_+ \psi_2}_{\sqrt{3}\,\psi_3} = \sqrt{3\times 2}\, \psi_3 = \sqrt{3!}\, \psi_3 \implies \psi_3 = \frac{(a_+)^3}{\sqrt{3!}} \psi_0$$

$$\boxed{\psi_n(x) = \frac{(a_+)^n}{\sqrt{n!}} \psi_0(x)}$$

- $$(\psi_m, a_+ \underbrace{a_- \psi_n}_{\sqrt{n}\,\psi_{n-1}}) = \sqrt{n}\,(\psi_m, \underbrace{a_+ \psi_{n-1}}_{\sqrt{n}\,\psi_n}) = n\,(\psi_m, \psi_n)$$

$$(a_+ \underbrace{a_- \psi_m}_{\sqrt{m}\,\psi_{m-1}}, \psi_n) = \sqrt{m}\,(\underbrace{a_+ \psi_{m-1}}_{\sqrt{m}\,\psi_m}, \psi_n) = m\,(\psi_m, \psi_n)$$

$$\xrightarrow[\text{hermitianos}]{a_+ a_-} (m-n)(\psi_m, \psi_n) = 0$$

$$\Downarrow$$

$$(\psi_m, \psi_n) = 0 \quad (m \neq n)$$

(ortogonalidade dos autoestados)

Valores médios

$$\begin{cases} a_+ = \dfrac{-i}{\sqrt{2m\hbar\omega_0}}\, p + \sqrt{\dfrac{m\omega_0}{2\hbar}}\, x \\[2ex] a_- = \dfrac{i}{\sqrt{2m\hbar\omega_0}}\, p + \sqrt{\dfrac{m\omega_0}{2\hbar}}\, x \end{cases} \Longrightarrow \begin{cases} x = \sqrt{\dfrac{\hbar}{2m\omega_0}}\,(a_+ + a_-) \\[2ex] p = -i\sqrt{\dfrac{m\hbar\omega_0}{2}}\,(a_+ - a_-) \end{cases} \Longrightarrow \begin{cases} x^2 = \dfrac{\hbar}{2m\omega_0}\left(a_+a_- + a_-a_+ + a_+^2 + a_-^2\right) \\[2ex] p^2 = -\dfrac{m\hbar\omega_0}{2}\left(-a_+a_- - a_-a_+ + a_+^2 + a_-^2\right) \end{cases}$$

$$V = \frac{1}{2} m\omega_0^2 x^2 = \frac{1}{4}\hbar\omega_0 \left(a_+a_- + a_-a_+ + a_+^2 + a_-^2 \right)$$

$$\langle V \rangle_n = \frac{1}{4}\hbar\omega_0 \left\{ \underbrace{\left(\psi_n, a_+ \underbrace{a_-\psi_n}_{\sqrt{n}\,\psi_{n-1}}\right)}_{n\,\psi_n} + \underbrace{\left(\psi_n, a_- \underbrace{a_+\psi_n}_{\sqrt{n+1}\,\psi_{n+1}}\right)}_{(n+1)\,\psi_n} + \cancelto{0}{\left(\psi_n, \underbrace{a_+^2\psi_n}_{\sim\psi_{n+2}}\right)} + \cancelto{0}{\left(\psi_n, \underbrace{a_-^2\psi_n}_{\sim\psi_{n-2}}\right)} \right\} = \frac{1}{4}\hbar\omega_0 (2n+1)$$

$$\boxed{\langle V \rangle_n = \frac{\hbar\omega_0}{2}\left(n + \tfrac{1}{2}\right)}$$

$$T = \frac{p^2}{2m} = -\frac{\hbar\omega_0}{4}\left(-a_+a_- - a_-a_+ + a_+^2 + a_-^2\right)$$

$$\langle T \rangle_n = -\frac{1}{4}\hbar\omega_0 \left\{ -\underbrace{(\psi_n, a_+a_-\psi_n)}_{n} - \underbrace{(\psi_n, a_-a_+\psi_n)}_{n+1} + \cancelto{0}{(\psi_n, a_+^2\psi_n)} + \cancelto{0}{(\psi_n, a_-^2\psi_n)} \right\} = \frac{\hbar\omega_0}{2}\left(n + \tfrac{1}{2}\right)$$

$$\boxed{\langle T \rangle_n = \frac{\hbar\omega_0}{2}\left(n + \tfrac{1}{2}\right)} \qquad \boxed{\langle V \rangle_n = \langle T \rangle_n = \frac{\langle E \rangle_n}{2}}$$

$$\langle x \rangle_n = (\psi_n, x\psi_n) = \sqrt{\frac{\hbar}{2m\omega_0}}\,(\psi_n, (a_+ + a)\psi_n) = \sqrt{\frac{\hbar}{2m\omega_0}}\left\{ \cancel{(\psi_n, \underbrace{a_+\psi_n}_{\sim\psi_{n+1}})}^{0} + \cancel{(\psi_n, \underbrace{a_-\psi_n}_{\psi_{n-1}})}^{0} \right\} = 0$$

$$\langle x^2 \rangle_n = \frac{2\langle V \rangle_n}{m\omega_0^2} \quad \frac{\hbar}{m\omega_0}(n + 1/2) = (\Delta x)_n^2$$

$$\langle p_n \rangle = (\psi_n, p\psi_n) = -i\sqrt{\frac{m\hbar\omega_0}{2}}\,(\psi_n, (a_+ - a_-)\psi_n) = -i\sqrt{\frac{m\hbar\omega_0}{2}}\left\{ \cancel{(\psi_n, a_+\psi_n)}^{0} - \cancel{(\psi_n, a_-\psi_n)}^{0} \right\} = 0$$

$$\langle p^2 \rangle_n = 2m\langle T \rangle_n = m\hbar\omega_0(n + 1/2) = (\Delta p)_n^2$$

$$\boxed{(\Delta x)_n (\Delta p)_n = \hbar(n + 1/2)} \qquad (\Delta x \Delta p)_{min} = \hbar/2$$

Exemplo

estado inicial : $\Psi(x,0) = \frac{1}{\sqrt{2}}\psi_0(x) + \frac{1}{\sqrt{2}}\psi_1(x)$

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\psi_0(x)\, e^{-iE_0/\hbar\, t} + \frac{1}{\sqrt{2}}\psi_1(x)\, e^{-i\frac{E_1}{\hbar}t}$$

$$\left\{ \begin{array}{ll} E_0 = \hbar\omega_0 & E_1 = \frac{3}{2}\hbar\omega_0 \\ E_1 - E_0 = \hbar\omega_0 & \end{array} \right.$$

$$\langle x\rangle_t = \left(\Psi(x,t), x\Psi(x,t)\right) = \frac{1}{2}\cancelto{0}{(\psi_0, x\psi_0)} + \frac{1}{2}(\psi_1, x\psi_0)\underbrace{e^{i\frac{(E_1-E_0)}{\hbar}t}}_{e^{i\omega_0 t}} + \frac{1}{2}(\psi_0, x\psi_1)\underbrace{e^{-i\frac{(E_1-E_0)}{\hbar}t}}_{e^{-i\omega_0 t}} + \frac{1}{2}\cancelto{0}{(\psi_1, x\psi_1)}$$

$$x = \sqrt{\frac{\hbar}{2m\omega_0}}(a_+ + a_-)$$

$$= \frac{1}{2}\cdot\sqrt{\frac{\hbar}{2m\omega_0}}\, e^{i\omega_0 t} + \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega_0}}\, e^{-i\omega_0 t}$$

$$\left\{ \begin{array}{l} a_+\psi_0 = \psi_1 \\ a_-\psi_1 = \psi_0 \end{array} \right.$$

$$\boxed{\langle x\rangle_t = \sqrt{\frac{\hbar}{2m\omega_0}}\cos\omega_0 t}$$

$$\frac{d\langle x\rangle_t}{dt} = \frac{\langle p\rangle_t}{m} \Longrightarrow \boxed{\langle p\rangle_t = m\frac{d\langle x\rangle_t}{dt} = -\sqrt{\frac{m\hbar\omega_0}{2}}\,\mathrm{sen}\,\omega_0 t}$$

Exercício

$$\Psi(x,0) = \frac{1}{\sqrt{8}}\,\psi_0(x) + \frac{1}{\sqrt{2}}\,\psi_1(x) + A\,\psi_2(x)$$

$\psi_n \to$ autoestados de energia do oscilador

(estado inicial)

Determine :

a) o valor de A ;

b) as possíveis energias e as respectivas probabilidades

c) os valores médios de x e p , em qualquer instante t .