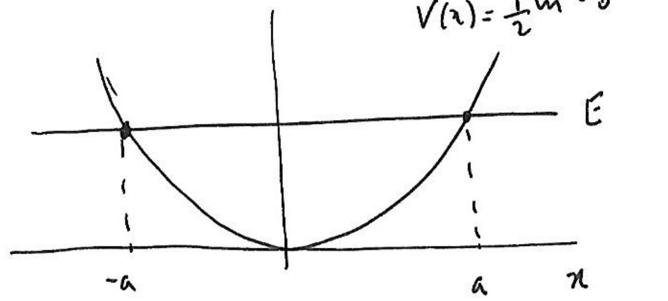
# Oscilador harmônico

- estados estacionários

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \underbrace{\frac{1}{2} m\omega_0^2 x^2}_{\text{função par}} \psi = E\psi$$

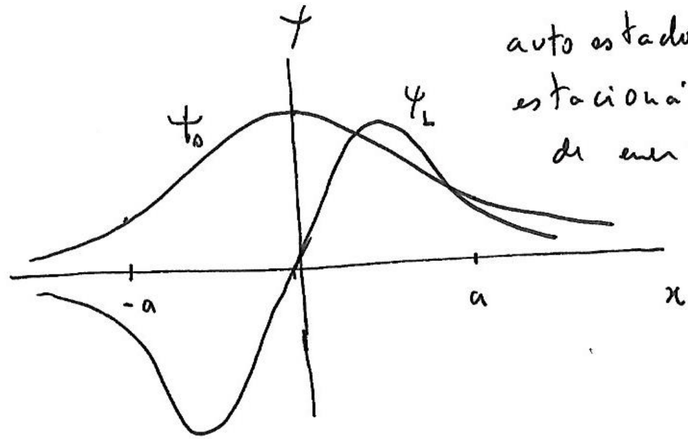



$-a$ e $+a$

ptos. de retrocesso clássico

limite assintótico

$$\begin{cases} x \to \pm\infty \quad \Rightarrow \quad \dfrac{d^2\psi_\infty}{dx^2} = \left(\dfrac{m\omega_0}{\hbar}\right)^2 x^2 \psi_\infty \\ \\ \psi_\infty \sim \pm e^{-\alpha x^2/2} \quad (\alpha = m\omega_0/\hbar) \end{cases}$$




auto estados estacionários de energia

$V(x)$ - par $\Rightarrow$ $\psi(x)$ - paridade definida (par ou ímpar)

$$\psi(x) = \underbrace{H(x)}_{\text{paridade definida}}\, e^{-\alpha \overbrace{x^2}^{\text{par}}/2}$$

- <u>estado fundamental</u>

$$H_0 = \overset{\text{cte}}{A_0} \ (\text{par}) \Rightarrow \begin{cases} \psi_0 = A_0 e^{-\alpha x^2/2} \qquad (\alpha = m\omega_0/\hbar) \\ \psi_0' = -\alpha x \psi_0 \Rightarrow \psi_0'' = -\alpha \psi_0 - \alpha x \psi_0' = \left[(\alpha x)^2 - \alpha\right] \psi_0 \end{cases}$$

$$\left[\cancel{(\alpha x)^2} - \alpha\right] \psi_0 = \cancel{(\alpha x)^2} \psi_0 - 2 E_0 \frac{m}{\hbar^2} \psi_0 \Rightarrow \boxed{E_0 = \hbar\omega_0/2 = \left(0 + \tfrac{1}{2}\right)\hbar\omega_0}$$

$$\boxed{\psi_0 = A_0 e^{-\alpha x^2/2}}$$

- <u>1º estado excitado</u>

$$H_1 = A x \ (\text{impar}) \Rightarrow \begin{cases} \psi_1 = A x e^{-\alpha x^2/2} \Rightarrow \psi_1' = A e^{-\frac{\alpha}{2}x^2} - \underbrace{\alpha x^2 A e^{-\alpha x^2/2}}_{(\alpha x)\psi_1} \\ \psi_1'' = -\alpha x A e^{-\alpha x^2/2} - \alpha \psi_1 - \underbrace{\alpha x \psi_1'}_{-\alpha \underbrace{x A e^{-\frac{\alpha}{2}x^2}}_{\psi_1} + (\alpha x)^2 \psi_1} \\ \quad = \left[(\alpha x)^2 - 3\alpha\right] \psi_1 \end{cases}$$

$$\left[(\alpha x)^2 - 3\alpha\right] \psi_1 = (\alpha x)^2 \psi_1 - 2 E_1 \frac{m}{\hbar^2} \psi_1 \Rightarrow \boxed{E_1 = \frac{3}{2}\hbar\omega_0 = \left(1 + \frac{1}{2}\right)\hbar\omega_0}$$

$$\boxed{\psi_1 = A_1 2x e^{-\alpha x^2/2}}$$

- <u>2º estado excitado</u>

$$H_2 = Ax^2 + B \ (\text{par}) \Rightarrow \begin{cases} \psi_2 = (Ax^2+B)\, e^{-\frac{\alpha}{2}x^2} \\ \psi_2' = \left[2Ax - \alpha x(Ax^2+B)\right] e^{-\frac{\alpha}{2}x^2} \\ \psi_2'' = \left\{2A - 3\alpha Ax^2 - \alpha B - (\alpha x)\left[2Ax - \alpha x(Ax^2+B)\right]\right\} e^{-\frac{\alpha}{2}x^2} \\ \quad = \left[2A - 5\alpha Ax^2 - \alpha B + (\alpha x)^2 (Ax^2+B)\right] e^{-\frac{\alpha}{2}x^2} \end{cases}$$

$$2A - 5\alpha Ax^2 - \alpha B + \cancel{(\alpha x)^2(Ax^2+B)} = \cancel{(\alpha x)^2(Ax^2+B)} - \overbrace{2E_2\frac{m}{\hbar}}^{5\alpha}(Ax^2+B)$$

$$\begin{cases} -5\alpha = -2E_2\dfrac{m}{\hbar} \Rightarrow \boxed{E_2 = \dfrac{5}{2}\hbar\omega_0 = \left(2+\dfrac{1}{2}\right)\hbar\omega_0} \\ 2A - \alpha B = -5\alpha \Rightarrow A = -2\alpha B \end{cases}$$

$$\psi_2 = (-2\alpha x^2 + 1)B\, e^{-\frac{\alpha x^2}{2}} \Rightarrow \boxed{\psi_2 = A_2(4\alpha x^2 - 2)\, e^{-\frac{\alpha x^2}{2}}}$$

## $3^{\underline{o}}$ estado excitado

$$H_3 = Ax^2 + Bx \text{ (impar)} \Longrightarrow \begin{cases} \psi_3 = (Ax^3 + Bx)\, e^{-\alpha x^2/2} \\ \psi_3' = \left[3Ax^2 + B - \alpha x (Ax^3 + Bx)\right] e^{-\alpha x^2/2} \\ \psi_3'' = \left\{ 6Ax - 4\alpha A x^3 - 2\alpha B x - \alpha x \left[3Ax^2 + B - \alpha A x^4 - \alpha B x^2\right] \right\} e^{-\alpha x^2/2} \end{cases}$$

$$6Ax - 7\alpha A x^3 - 3\alpha B x + \underbrace{(\alpha x)^2 A x^3 + (\alpha x)^2 B x}_{\cancel{(\alpha x)^2 (Ax^3 + Bx)}} = \cancel{(\alpha x)^2 (Ax^3 + Bx)} - \overbrace{\frac{2 E_3 m}{\hbar^2}}^{7\alpha} (Ax^3 + Bx)$$

$$-7\alpha = -\frac{2 E_3 m}{\hbar} \Longrightarrow \boxed{E_3 = \frac{7}{2} \hbar \omega_0 = \left(3 + \frac{1}{2}\right) \hbar \omega_0}$$

$$6A - 3\alpha B = -7\alpha B \Longrightarrow A = -\frac{2}{3} \alpha B$$

$$\psi_3 = \left(-\frac{2}{3}\alpha x^3 + x\right) B\, e^{-\alpha x^2/2} \Longrightarrow$$

$$\psi_3 = A_3 \left[ 8 (\sqrt{\alpha}\, x)^3 - 12 \sqrt{\alpha}\, x \right] e^{-\alpha x^2/2}$$

# Relação com os polinômios de Hermite

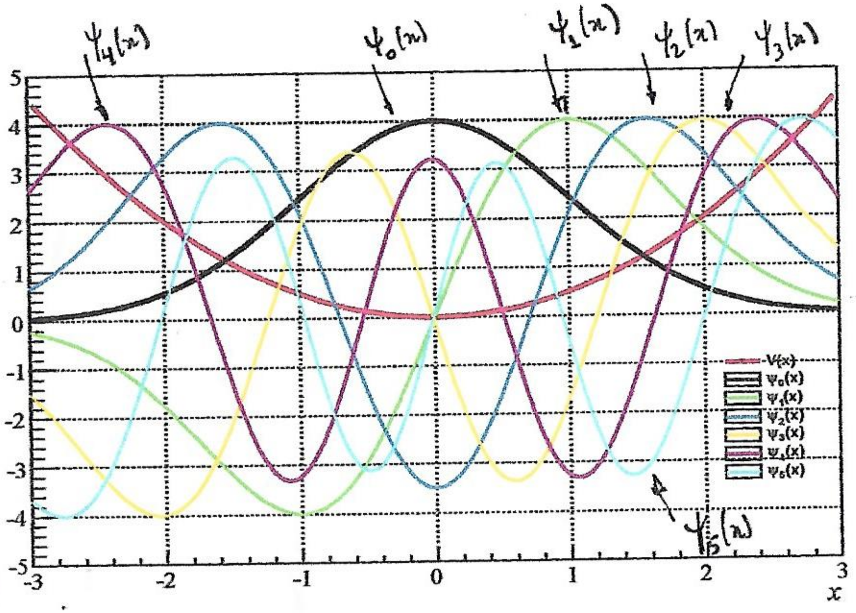
$$\psi(x) = H(x)\, e^{-\alpha x^2/2} \qquad (\alpha = m\omega_0/\hbar)$$

$$\Rightarrow \begin{cases} \psi' = (-\alpha x H + H')\, e^{-\alpha x^2/2} \\ \psi'' = \left[-\alpha H - \alpha x H' + H'' - (\alpha x)(-\alpha x H + H')\right] e^{-\alpha x^2/2} = H'' - 2\alpha x H' - \alpha H + (\alpha x)^2 H \end{cases}$$

$$H'' - 2\alpha x H' + \alpha(\varepsilon - 1) H = 0 \qquad \varepsilon = E/_{\hbar\omega_0/2}$$

$$\xi = \sqrt{\alpha}\, x \Rightarrow \begin{cases} \dfrac{d^2H}{dx^2} = \alpha \dfrac{d^2H}{d\xi^2} \\ -2\alpha x \dfrac{dH}{dx} = -2\alpha\, \xi \dfrac{dH}{d\xi} \end{cases} \Rightarrow \boxed{H'' - 2\xi H' + (\varepsilon - 1) H(\xi) = 0} \quad \text{eq. dif. de Hermite}$$

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \left(\frac{1}{2^n n!}\right)^{1/2} \exp\left(\frac{-m\omega}{2\hbar} x^2\right) H_n\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right)$$




| Primeiros Polinômios de Hermite |
|---|
| $H_0(\xi) = 1$ |
| $H_1(\xi) = 2\xi$ |
| $H_2(\xi) = 4\xi^2 - 2$ |
| $H_3(\xi) = 8\xi^3 - 12\xi$ |
| $H_4(\xi) = 16\xi^4 - 48\xi^2 + 12$ |
| $H_5(\xi) = 32\xi^5 - 160\xi^3 + 120\xi$ |
| $H_6(\xi) = 64\xi^6 - 480\xi^4 + 720\xi^2 - 120$ |