Momentum angular

$$\vec{L} = \vec{r}\times\vec{p} = -i\hbar\,\vec{r}\times\nabla \qquad\qquad \vec{p} = -i\hbar\nabla$$

$$\begin{cases} p_x = -i\hbar\,\partial/\partial x & [p_x,p_y]=0 \\ p_y = -i\hbar\,\partial/\partial y & [p_x,p_z]=0 \\ p_z = -i\hbar\,\partial/\partial z & [p_y,p_z]=0 \end{cases}$$

$$\begin{cases} [x,p_x]=i\hbar & [x,p_y]=0 & [x,p_z]=0 \\ [y,p_x]=0 & [y,p_y]=i\hbar & [y,p_z]=0 \\ [z,p_x]=0 & [z,p_y]=0 & [z,p_z]=i\hbar \end{cases}$$

$$\begin{cases} L_x = y p_z - z p_y = -i\hbar\left(y\dfrac{\partial}{\partial z} - z\dfrac{\partial}{\partial y}\right) \\ L_y = z p_x - x p_z = -i\hbar\left(z\dfrac{\partial}{\partial x} - x\dfrac{\partial}{\partial z}\right) \\ L_z = x p_y - y p_x = -i\hbar\left(x\dfrac{\partial}{\partial y} - y\dfrac{\partial}{\partial x}\right) \end{cases}$$

$$[L_x,L_y] = L_x L_y - L_y L_x$$

$$L_x L_y = (y p_z - z p_y)(z p_x - x p_z) = y p_z z p_x - \cancel{y p_z x p_z} - \cancel{z p_y z p_x} + \overbrace{z p_y x p_z}^{x z p_z p_y}$$

$$- L_y L_x = -(z p_x - x p_z)(y p_z - z p_y) = -z p_x y p_z + z p_x z p_y + x p_z y p_z - x p_z z p_y$$

$$= -y z p_z p_x + \cancel{z p_y z p_x} + \cancel{y p_z x p_z} - x p_z z p_y$$

$$[L_x,L_y] = y\,(\underbrace{p_z z - z p_z}_{[p_z,z]=-i\hbar})\,p_x + x\,(\underbrace{z p_z - p_z z}_{[z,p_z]=i\hbar})\,p_y = i\hbar\,(\underbrace{x p_y - y p_x}_{L_z})$$

$$\boxed{[L_x,L_y] = i\hbar\,L_z}$$

$$\boxed{[L_\alpha,L_\beta] = i\hbar\,\varepsilon_{\alpha\beta\gamma}L_\gamma}$$

$$\boxed{L^2 = L_x^2 + L_y^2 + L_z^2}$$

$$[L^2, L_x] = [L_x^2 + L_y^2 + L_z^2, L_x] = \cancelto{0}{[L_x^2, L_x]} + [L_y^2, L_x] + [L_z^2, L_x]$$

$$= [L_y L_y, L_x] + [L_z L_z, L_x]$$

$$= L_y L_y L_x - L_x L_y L_y + L_z L_z L_x - L_x L_z L_z +$$

$$- L_y L_x L_y + L_y L_x L_y - L_z L_x L_z + L_z L_x L_z$$

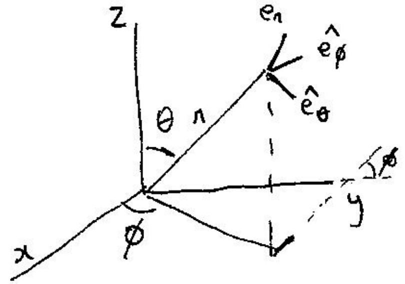
$$= L_y \underbrace{[L_y, L_x]}_{-i\hbar L_z} + \underbrace{[L_y, L_x]}_{-i\hbar L_z} L_y + L_z \underbrace{[L_z, L_x]}_{i\hbar L_y} + \underbrace{[L_z, L_x]}_{i\hbar L_y} L_z$$

$$= -i\hbar \cancel{L_y L_z} - i\hbar \cancel{L_z L_y} + i\hbar \cancel{L_z L_y} + i\hbar \cancel{L_y L_z} = 0$$

$$\boxed{[L^2, L_x] = [L^2, L_y] = [L^2, L_z] = 0}$$

Momentum angular em coordenadas esféricas

$$\begin{cases} \vec L = \vec r \times \vec p = -i\hbar\, \vec r \times \nabla \\[2mm] \vec r = r\,\hat e_r = x\,\hat\imath + y\,\hat\jmath + z\,\hat k \\[2mm] \nabla = \left(\dfrac{\partial}{\partial r},\ \dfrac{1}{r}\dfrac{\partial}{\partial\theta},\ \dfrac{1}{r\,\text{sen}\,\theta}\dfrac{\partial}{\partial\phi}\right) \end{cases}$$



$$\begin{cases} x = r\,\text{sen}\,\theta\cos\phi \\ y = r\,\text{sen}\,\theta\,\text{sen}\,\phi \\ z = r\cos\theta \end{cases}$$

$$\hat e_\phi = -\text{sen}\,\phi\,\hat\imath + \cos\phi\,\hat\jmath \implies \frac{\partial\hat e_\phi}{\partial\phi} = -\cos\phi\,\hat\imath - \text{sen}\,\phi\,\hat\jmath$$

$$\hat e_r = \frac{\partial\vec r}{\partial r} = \text{sen}\,\theta\cos\phi\,\hat\imath + \text{sen}\,\theta\,\text{sen}\,\phi\,\hat\jmath + \cos\theta\,\hat k \implies \frac{\partial\hat e_r}{\partial\phi} = -\text{sen}\,\theta\,\text{sen}\,\phi\,\hat\imath + \text{sen}\,\theta\cos\phi\,\hat\jmath = \text{sen}\,\theta\,\hat e_\phi$$

$$e_\theta = \frac{\partial\hat e_r}{\partial\theta} = \cos\theta\cos\phi\,\hat\imath + \cos\theta\,\text{sen}\,\phi\,\hat\jmath - \text{sen}\,\theta\,\hat k \implies \begin{cases} \dfrac{\partial\hat e_\theta}{\partial\theta} = -\text{sen}\,\theta\cos\phi\,\hat\imath - \text{sen}\,\theta\,\text{sen}\,\phi\,\hat\jmath - \cos\theta\,\hat k = -\hat e_r \\[3mm] \dfrac{\partial\hat e_\theta}{\partial\phi} = -\cos\theta\,\text{sen}\,\phi\,\hat\imath + \cos\theta\cos\phi\,\hat\jmath - \cos\theta\,\hat k \end{cases}$$

$$\vec r \times \nabla = \frac{\vec L}{(-i\hbar)} = \vec l = \begin{pmatrix} \hat e_r & \hat e_\theta & \hat e_\phi \\ r & 0 & 0 \\ \partial/\partial r & \dfrac{1}{r}\dfrac{\partial}{\partial\theta} & \dfrac{1}{r\,\text{sen}\,\theta}\dfrac{\partial}{\partial\phi} \end{pmatrix} = -\frac{\hat e_\theta}{\text{sen}\,\theta}\frac{\partial}{\partial\phi} + \hat e_\phi\frac{\partial}{\partial\theta}$$

$$\vec l\cdot\vec l = l^2 = -\frac{L^2}{\hbar^2} = -\frac{\hat e_\theta}{\text{sen}\,\theta}\frac{\partial}{\partial\phi}\cdot\left(\frac{-\hat e_\theta}{\text{sen}\,\theta}\frac{\partial}{\partial\phi}\right) - \frac{\hat e_\theta}{\text{sen}\,\theta}\frac{\partial}{\partial\phi}\cdot\left(\hat e_\phi\frac{\partial}{\partial\theta}\right) + \hat e_\phi\frac{\partial}{\partial\theta}\cdot\left(\frac{-\hat e_\theta}{\text{sen}\,\theta}\frac{\partial}{\partial\phi}\right) + \hat e_\theta\frac{\partial}{\partial\theta}\cdot\left(\hat e_\theta\frac{\partial}{\partial\theta}\right)$$

$$= \frac{1}{\text{sen}^2\theta}\frac{\partial^2}{\partial\phi^2} - \frac{1}{\text{sen}\,\theta}\underbrace{\left(\hat e_\theta\cdot\frac{\partial\hat e_\phi}{\partial\phi}\right)}_{-\cos\theta}\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} \implies \boxed{l^2 = -\frac{L^2}{\hbar^2} = \frac{\partial^2}{\partial\theta^2} + \text{cotg}\,\theta\,\frac{\partial}{\partial\theta} + \frac{1}{\text{sen}^2\theta}\frac{\partial^2}{\partial\phi^2}}$$

Laplaciano em coordenadas esféricas

$$\nabla^2 = \nabla\cdot\nabla = \hat{e}_r \frac{\partial}{\partial r}\cdot\left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{r \,\text{sen}\,\theta}\frac{\partial}{\partial \phi}\right) + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\cdot\left(\sim\right) + \hat{e}_\phi \frac{1}{r\,\text{sen}\,\theta}\frac{\partial}{\partial \phi}\cdot\left(\sim\right)$$

$$\left(\hat{e}_\theta \cdot \underbrace{\frac{\partial \hat{e}_r}{\partial \theta}}_{\hat{e}_\theta} = 1\right)$$

$$\begin{cases} \hat{e}_\phi \cdot \dfrac{\partial \hat{e}_r}{\partial \phi} = \text{sen}\,\theta \\ \hat{e}_\phi \cdot \dfrac{\partial \hat{e}_\theta}{\partial \phi} = \cos\theta\,\text{sen}^2\phi + \cos\theta\cos^2\phi = \cos\theta \end{cases}$$

$$= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\,\text{cotg}\,\theta\frac{\partial}{\partial \theta} + \frac{1}{r^2\,\text{sen}^2\theta}\frac{\partial^2}{\partial \phi^2}$$

$$\boxed{\nabla^2 = \underbrace{\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)}_{\nabla_r^2} + \underbrace{\frac{1}{r^2}\left[\frac{\partial^2}{\partial \theta^2} + \text{cotg}\,\theta\frac{\partial}{\partial \theta} + \frac{1}{\text{sen}^2\theta}\frac{\partial^2}{\partial \phi^2}\right]}_{\nabla_{\theta,\phi}^2 = \frac{l^2(\theta,\phi)}{r^2} = -\frac{L^2(\theta,\phi)}{\hbar^2 r^2}}}$$

$$\boxed{\nabla^2(r,\theta,\phi) = \nabla_r^2 + \nabla_{\theta,\phi}^2 = \nabla_r^2 + \frac{l^2(\theta,\phi)}{r^2} = \nabla_r^2 - \frac{L^2(\theta,\phi)}{\hbar^2 r^2}}$$

$$\begin{cases} \left[l^2, \nabla_r^2\right] = 0 \quad \Rightarrow \quad \left[\nabla^2, L^2\right] = 0 \\ \left[l_z, \nabla_r^2\right] = 0 \quad \left[l_z, l^2\right] = 0 \quad \Rightarrow \quad \left[L_z, \nabla^2\right] = 0 \end{cases}$$

∇^2, L^2 e L_z possuem autofunções simultâneas