Conservação de probabilidade

$$\begin{cases} \left[-\dfrac{\hbar^2}{2m}\nabla^2 + V(\vec{r})\right]\Psi(\vec{r},t) = i\hbar\dfrac{\partial\Psi}{\partial t} \\ \rho(\vec{r},t) = \Psi^*(\vec{r},t)\,\Psi(\vec{r},t) = |\Psi(\vec{r},t)|^2 \end{cases}$$

$$\begin{cases} \dfrac{\partial\Psi}{\partial t} = -\dfrac{i}{\hbar}\left[-\dfrac{\hbar^2}{2m}\nabla^2 + V\right]\Psi \\ \dfrac{\partial\Psi^*}{\partial t} = \dfrac{i}{\hbar}\left[-\dfrac{\hbar^2}{2m}\nabla^2 + V\right]\Psi^* \end{cases}$$

$$\frac{\partial\rho}{\partial t} = \left(\frac{\partial\Psi^*}{\partial t}\right)\Psi + \Psi^*\left(\frac{\partial\Psi}{\partial t}\right) = \frac{i}{\hbar}\left[\left(-\frac{\hbar^2}{2m}\nabla^2 + \cancel{V}\right)\Psi^*\right]\Psi + \Psi^*\left(\frac{-i}{\hbar}\right)\left(-\frac{\hbar^2}{2m}\nabla^2 + \cancel{V}\right)\Psi$$

$$= \frac{i\hbar}{2m}\left[\Psi^*(\nabla^2\Psi) - \Psi(\nabla^2\Psi^*)\right] = \left[\nabla\cdot(\Psi^*\nabla\Psi) - \cancel{(\nabla\Psi^*)(\nabla\Psi)} - \nabla\cdot(\Psi\nabla\Psi^*) + \cancel{(\nabla\Psi)(\nabla\Psi^*)}\right]\frac{i\hbar}{2m}$$

$$= -\nabla\cdot\left[\underbrace{+\frac{i\hbar}{2m}\left(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi\right)}_{\vec{J}}\right]$$

$$\boxed{\vec{J} = \frac{i\hbar}{2m}\left(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi\right)}$$

$\Downarrow$

$\boxed{\dfrac{\partial\rho}{\partial t} + \nabla\cdot\vec{J} = 0}$ eq. de continuidade $\Longrightarrow$ $\boxed{\dfrac{d}{dt}\displaystyle\int_V \rho\, dV = -\oint_S \vec{J}\cdot d\vec{S}}$

$$\int_V \frac{\partial\rho}{\partial t}\,dV = -\int_V (\nabla\cdot\vec{J})\,dV = -\oint_S \vec{J}\cdot d\vec{S} = \frac{d}{dt}\int_V \rho\,dV$$

$V\to\infty \Longrightarrow \displaystyle\oint_S \vec{J}\cdot d\vec{S} = 0 \Longrightarrow \boxed{\displaystyle\int_{V\to\infty} \rho\,dV = \text{cte}}$ (normalização não depende do tempo) partícula livre ~~(unitidade~~

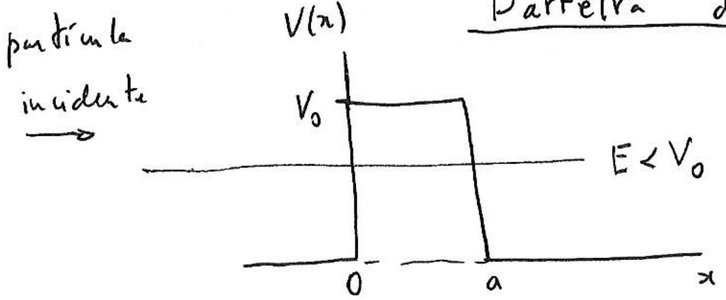
$$\begin{cases} \psi(\vec r,t) = A\, e^{i(\vec p\cdot\vec r - Et)/\hbar} & \Longrightarrow \nabla\psi = i\dfrac{\vec p}{\hbar}\psi \Longrightarrow \psi^*\nabla\psi = i\dfrac{\vec p}{\hbar}|A|^2 \\ \psi^*(\vec r,t) = A^*\, e^{-i(\vec p\cdot\vec r - Et)/\hbar} & \Longrightarrow \nabla\psi^* = -i\dfrac{\vec p}{\hbar}\psi^* \Longrightarrow \psi\nabla\psi^* = -i\dfrac{\vec p}{\hbar}|A|^2 \end{cases}$$

$$\vec J = \frac{i\hbar}{2m}\left(-i\frac{\vec p}{\hbar}|A|^2 - i\frac{\vec p}{\hbar}|A|^2\right) = \frac{\vec p}{m}|\vec A|^2$$

- $\psi(\vec r,t) = A\, e^{-i\frac{\vec p}{\hbar}\cdot\vec r}\, e^{-i\frac{E}{\hbar}t} \Longrightarrow \vec J = -\dfrac{\vec p}{m}|\vec A|^2$

- $\psi(\vec r,t) = A\, e^{i\vec k\cdot\vec r}\, e^{-i\frac{E}{\hbar}t} \Longrightarrow \vec J = \dfrac{\hbar\vec k}{m}|A|^2$

Barreira de potencial



$$V(x) = \begin{cases} 0 & x < 0 \quad \text{(região I)} \\ V_0 & 0 < x < a \quad \text{(região II)} \\ 0 & x > a \quad \text{(região III)} \end{cases}$$

$E < V_0$

região I: $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_I = E\psi_I \Rightarrow \frac{d^2}{dx^2}\psi_I + \overbrace{\frac{2mE}{\hbar^2}}^{k_0^2>0}\psi_I = 0$

$$\boxed{\psi_I = \underbrace{A e^{ik_0x}}_{\psi_{inc.}} + \underbrace{B e^{-ik_0x}}_{\psi_{refl.}}} \Rightarrow \boxed{J_{inc} = \frac{\hbar k_0}{m}|A|^2} \quad \boxed{J_{refl.} = -\frac{\hbar k_0}{m}|B|^2}$$

região II: $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_{II} + V_0\psi_{II} = E\psi_{II} \Rightarrow \frac{d^2\psi_{II}}{dx^2} - \overbrace{\frac{2m(V_0-E)}{\hbar^2}}^{\rho^2>0}\psi_{II} = 0$

$$\boxed{\psi_{II} = Ce^{\rho x} + De^{-\rho x}}$$

região III: $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_{III} = E\psi_{III} \Rightarrow \frac{d^2\psi_{III}}{dx^2} + \overbrace{\frac{2mE}{\hbar^2}}^{k_0^2>0}\psi_{III} = 0$

$$\boxed{\psi_{III} = Fe^{ik_0x}} \Rightarrow \boxed{J_{trans} = \frac{\hbar k_0}{m}|F|^2}$$

coeficientes de

reflexão $r = \left|\frac{J_{refl.}}{J_{inc}}\right| = \left|\frac{B}{A}\right|^2$

transmissão $t = \left|\frac{J_{trans.}}{J_{inc}}\right| = \left|\frac{F}{A}\right|^2$

condições de contorno

$$\begin{cases} \psi_I(0) = \psi_{II}(0) \implies A + B = C + D \\ \psi'_I(0) = \psi'_{II}(0) \implies ik_0(A-B) = \rho(C-D) \\ \psi_{II}(a) = \psi_{III}(a) \implies Ce^{\rho a} + De^{-\rho a} = Fe^{ik_0 a} \\ \psi'_{II}(a) = \psi'_{III}(a) \implies \rho(Ce^{\rho a} - De^{-\rho a}) = ikFe^{ik_0 a} \end{cases} \implies \begin{cases} 2Ce^{\rho a} = \left(1 + \frac{ik_0}{\rho}\right)Fe^{ik_0 a} \\ 2De^{-\rho a} = \left(1 - \frac{ik_0}{\rho}\right)Fe^{ik_0 a} \end{cases}$$

$$\begin{cases} 2\rho(C+D) = \left[(\rho + ik_0)e^{-\rho a} + (\rho - ik_0)e^{\rho a}\right]Fe^{ik_0 a} = 2\rho(A+B) \\ 2\rho(C-D) = \left[(\rho + ik_0)e^{-\rho a} + (\rho - ik_0)e^{\rho a}\right]Fe^{ik_0 a} = 2ik_0(A-B) \end{cases}$$

$$4\frac{A}{F}e^{-ik_0 a} = e^{-\rho a}(\rho + ik)\overbrace{\left(\frac{1}{\rho} + \frac{1}{ik_0}\right)}^{(\rho+ik_0)/i\rho k_0} + e^{\rho a}(\rho - ik)\overbrace{\left(\frac{1}{\rho} - \frac{1}{ik_0}\right)}^{-(\rho-ik_0)/i\rho k_0}$$

$$4\frac{A}{F}(i\rho k_0)e^{-ik_0 a} = e^{-\rho a}(\rho^2 + 2i\rho k_0 - k_0^2) - e^{\rho a}(\rho^2 - 2i\rho k_0 - k_0^2)$$

$$= 2i\rho k_0 \underbrace{(e^{\rho a} + e^{-\rho a})}_{2\cosh \rho a} - (\rho^2 - k_0^2)\underbrace{(e^{\rho a} - e^{-\rho a})}_{2\operatorname{senh} \rho a}$$

$$\frac{A}{F}e^{-ik_0 a} = \cosh \rho a - \frac{(\rho^2 - k_0^2)}{2i\rho k_0}\operatorname{senh}\rho a \implies \left|\frac{A}{F}\right|^2 = \underbrace{\cosh^2 \rho a}_{1+\operatorname{senh}^2\rho a} + \frac{(\rho^2 - k_0^2)^2}{4\rho^2 k_0^2}\operatorname{senh}^2 \rho a$$

$$\left|\frac{A}{F}\right|^2 = 1 + \underbrace{\left[1 + \frac{(\rho^2 - k_0^2)^2}{4\rho^2 k_0^2}\right]}_{\frac{(\rho^2+k_0^2)^2}{4\rho^2 k_0^2}}\operatorname{senh}^2 \rho a = 1 + \frac{(\rho^2 + k_0^2)^2}{4\rho^2 k_0^2}\operatorname{senh}^2 \rho a$$

$$\left|\frac{A}{F}\right|^2 = 1 + \frac{(\rho^2 + k_0^2)^2}{4\rho^2 k_0^2} \operatorname{senh}^2 \rho a$$

$$\begin{cases} k_0^2 = 2mE/\hbar^2 \\ \rho^2 = 2m(V_0 - E)/\hbar^2 \end{cases} \Rightarrow \begin{cases} (\rho^2 + k_0^2) = \left(\dfrac{2mV_0}{\hbar^2}\right)^2 \\ 4\rho^2 k_0^2 = 4\left(\dfrac{2m}{\hbar^2}\right)^2 E(V_0 - E) \end{cases}$$

$$= 1 + \frac{V_0^2}{4E(V_0-E)} \operatorname{senh}^2 \rho a$$

$$\boxed{t = \left[1 + \frac{V_0^2}{4E(V_0-E)} \operatorname{senh}^2 \rho a\right]^{-1}}$$

$$\rho a = \frac{\sqrt{2m(V_0-E)}}{\hbar}\, a \gg 1$$ (barreira muito alta ou larga) $\Rightarrow \operatorname{senh}^2 \rho a = \dfrac{e^{2\rho a}}{4}$

$$t = 16\,\frac{E(V_0-E)}{V_0^2}\, e^{-2\rho a} \ll 1$$ (apesar da pequena probabilidade, algumas partículas atravessam a barreira – <u>efeito túnel</u>)