Composição de momentos angulares

$s_1 = 1/2$, $s_2 = 1/2$ $\rightarrow$ j: $s_1 + s_2 = 1$ (m_j = $1, 0, -1$); $|s_1 - s_2| = 0$ (m_j = 0)

$|1,1\rangle$, $|1,0\rangle$, $|1,-1\rangle$ } autoestados de $j=1$ (tripleto)

$|0,0\rangle$ | autoestado de $j=0$ (singleto)

$$\begin{cases} |+\rangle_1 \ , \ |-\rangle_1 & \text{(autoestados de } s_1 = 1/2) \\ |+\rangle_2 \ , \ |-\rangle_2 & \text{(autoestados de } s_2 = 1/2) \end{cases}$$

autoestados de $j=1$ e $j=0$ podem ser expressos por combinações lineares de produtos
$|+\rangle_1 |-\rangle_2$, $|-\rangle_1 |+\rangle_2$, $|+\rangle_1 |+\rangle_2$, $|-\rangle_1 |-\rangle_2$ segundo

$$(\text{simétrica}) \begin{cases} |1,1\rangle = |+\rangle_1 |+\rangle_2 \\ |1,0\rangle = \frac{1}{\sqrt{2}} \left[|+\rangle_1 |-\rangle_2 + |-\rangle_1 |+\rangle_2 \right] \\ |1,-1\rangle = |-\rangle_1 |-\rangle_2 \end{cases}$$

$$|0,0\rangle = \frac{1}{\sqrt{2}} \left[|+\rangle_1 |-\rangle_2 - |-\rangle_1 |+\rangle_2 \right] \quad (\text{antisimétrica})$$

$$J^2\,|1,0\rangle = j(j+1)\,|1,0\rangle = 2\,|1,0\rangle = \frac{2}{\sqrt{2}}\Big[\,|+\rangle_1|-\rangle_2 + |-\rangle_1|+\rangle_2\Big]$$

$$\vec{J} = \vec{S}_1 + \vec{S}_2 \implies J^2 = S_1^2 + S_2^2 + 2\,\vec{S}_1\cdot\vec{S}_2 \qquad \vec{S}_1\cdot\vec{S}_2 = S_{1x}S_{2x} + S_{1y}S_{2y} + S_{1z}S_{2z}$$

$$\begin{cases} S_1^2\,|+\rangle_1|-\rangle_2 = \frac{3}{4}\,|+\rangle_1|-\rangle_2 = S_2^2\,|+\rangle_1|-\rangle_2 \\ S_1^2\,|-\rangle_1|+\rangle_2 = \frac{3}{4}\,|-\rangle_1|+\rangle_2 = S_2^2\,|-\rangle_1|+\rangle_2 \end{cases}$$

$$\begin{cases} S_{1x}S_{2x}\,|+\rangle_1|-\rangle_2 = S_{1x}|+\rangle_1\,S_{2x}|-\rangle_2 = \frac{1}{4}\underbrace{\begin{pmatrix}0&1\\1&0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_1}_{\begin{pmatrix}0\\1\end{pmatrix}_1}\underbrace{\begin{pmatrix}0&1\\1&0\end{pmatrix}_2\begin{pmatrix}0\\1\end{pmatrix}_2}_{\begin{pmatrix}1\\0\end{pmatrix}_2} = \frac{1}{4}\,|-\rangle_1|+\rangle_2 \\ S_{1y}S_{2y}\,|+\rangle_1|-\rangle_2 = \frac{1}{4}\underbrace{\begin{pmatrix}0&-i\\i&0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_1}_{i\begin{pmatrix}0\\1\end{pmatrix}_1}\underbrace{\begin{pmatrix}0&-i\\i&0\end{pmatrix}_2\begin{pmatrix}0\\1\end{pmatrix}_2}_{-i\begin{pmatrix}1\\0\end{pmatrix}} = \frac{1}{4}\,|-\rangle_1|+\rangle_2 \\ S_{1z}S_{2z}\,|+\rangle_1|-\rangle_2 = \frac{1}{4}\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_1}_{\begin{pmatrix}1\\0\end{pmatrix}_1}\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}_2\begin{pmatrix}0\\1\end{pmatrix}_2}_{\begin{pmatrix}0\\-1\end{pmatrix}} = -\frac{1}{4}\,|+\rangle_1|-\rangle_2 \end{cases}$$

$$\implies \boxed{\vec{S}_1\cdot\vec{S}_2\,|+\rangle_1|-\rangle_2 = -\frac{1}{4}\,|+\rangle_1|-\rangle_2 + \frac{1}{2}\,|-\rangle_1|+\rangle_2}$$

$$\begin{cases} S_{1x}S_{2x}\,|-\rangle_1|+\rangle_2 = \frac{1}{4}\,|+\rangle_1|-\rangle_2 \\ S_{1y}S_{2y}\,|-\rangle_1|+\rangle_2 = \frac{1}{4}\,|+\rangle_1|-\rangle_2 \\ S_{1z}S_{2z}\,|-\rangle_1|+\rangle_2 = -\frac{1}{4}\,|-\rangle_1|+\rangle_2 \end{cases} \implies \boxed{\vec{S}_1\cdot\vec{S}_2\,|-\rangle_1|+\rangle_2 = -\frac{1}{4}\,|-\rangle_1|+\rangle_2 + \frac{1}{2}\,|+\rangle_1|-\rangle_2}$$

$$J^2\,|1,0\rangle = \frac{1}{\sqrt{2}}\Big\{\underbrace{S_1^2}_{3/4}\big[|+\rangle_1|-\rangle_2 + |-\rangle_1|+\rangle_2\big] + \underbrace{S_2^2}_{3/4}\big[|+\rangle_1|-\rangle_2 + |-\rangle_1|+\rangle_2\big] + \underbrace{2\,\vec{S}_1\cdot\vec{S}_2}_{1/2}\big[|+\rangle_1|-\rangle_2 + |-\rangle_1|+\rangle_2\big]\Big\} = \underbrace{\left(\frac{3}{4} + \frac{3}{4} + \frac{1}{2}\right)}_{2}|1,0\rangle$$

Composição $1 \otimes 1/2$

$$|3/2, 3/2\rangle = |1,1\rangle\, |1/2, 1/2\rangle$$

$$J_\pm |j, m\rangle = \sqrt{j(j+1) - m(m\pm 1)}\; |j, m\pm 1\rangle$$

$$J_- |3/2, 3/2\rangle = \sqrt{3}\; |3/2, +1/2\rangle$$

$$= \left(J_{1-} |1,1\rangle\right) |1/2, 1/2\rangle + |1,1\rangle \left(J_{2-} |1/2, 1/2\rangle\right)$$

$$= \sqrt{2}\; |1,0\rangle\, |1/2, 1/2\rangle + |1,1\rangle\, |1/2, -1/2\rangle$$

$$\boxed{\begin{array}{l} \sqrt{3/2\,(3/2+1) - 3/2\,(3/2 - 1)} = \\ \sqrt{\frac{15}{4} - \frac{3}{4}} = \sqrt{\frac{12}{4}} = \sqrt{3} \end{array}}$$

$$\boxed{\begin{array}{l} \sqrt{1\,(1+1) - 1\,\cancel{(1-1)}} = \sqrt{2} \\ \sqrt{1/2\,\underbrace{(1/2+1)}_{3/2} - 1/2\,\underbrace{(1/2-1)}_{-1/2}} = \\ \sqrt{\frac{3}{4} + \frac{1}{4}} = 1 \end{array}}$$

$$|3/2, 1/2\rangle = \frac{1}{\sqrt{3}}\, |1,1\rangle\, |1/2, -1/2\rangle + \sqrt{\frac{2}{3}}\, |1,0\rangle\, |1/2, 1/2\rangle$$

$|j, m_j\rangle$ – combinações lineares de

$$\left\{ \begin{array}{l} |1,1\rangle\, |1/2, 1/2\rangle \\ |1,1\rangle\, |1/2, -1/2\rangle \\ |1,0\rangle\, |1/2, 1/2\rangle \\ |1,0\rangle\, |1/2, -1/2\rangle \\ |1,-1\rangle\, |1/2, 1/2\rangle \\ |1,-1\rangle\, |1/2, -1/2\rangle \end{array} \right.$$

$$|j, m_j\rangle \begin{cases} |3/2, 3/2\rangle,\ |3/2, 1/2\rangle,\ |3/2, -1/2\rangle,\ |3/2, -3/2\rangle \\ |1/2, 1/2\rangle,\ |1/2, -1/2\rangle \end{cases}$$

Coeficientes de Clebsch-Gordon

$$\begin{cases} |j, m_j\rangle = \displaystyle\sum_{m_1, m_2} C_{m_1, m_2} \, |j_1, m_1\rangle \, |j_2, m_2\rangle \quad (m_1 + m_2 = m_j) \\[2ex] |j_1, m_1\rangle \, |j_2, m_2\rangle = \displaystyle\sum_{j} C^*_{m_1, m_2} \, |j, m_j\rangle \quad (m_1 + m_2 = m_j) \end{cases}$$

coeficientes de Clebsch-Gordon

$$|3/2, 1/2\rangle = \sqrt{\frac{1}{3}} \, |1,1\rangle \, |1/2, -1/2\rangle + \sqrt{\frac{2}{3}} \, |1,0\rangle \, |1/2, 1/2\rangle$$

$$\begin{cases} |1,1\rangle \, |1/2, -1/2\rangle = \sqrt{\frac{1}{3}} \, |3/2, 1/2\rangle + \sqrt{\frac{2}{3}} \, |1/2, 1/2\rangle \\[2ex] |1,0\rangle \, |1/2, 1/2\rangle = \sqrt{\frac{2}{3}} \, |3/2, 1/2\rangle - \sqrt{\frac{1}{3}} \, |1/2, 1/2\rangle \end{cases}$$

1/2 × 1/2

		1
		+1
+1/2	+1/2	1

		1	0
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

		1
		−1
−1/2	−1/2	1

2 × 1/2

		5/2
		+5/2
+2	+1/2	1

		5/2	3/2
		+3/2	+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

		5/2	3/2
		+1/2	+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

		5/2	3/2
		−1/2	−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

		5/2	3/2
		−3/2	−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

		5/2
		−5/2
−2	−1/2	1

1 × 1/2

		3/2
		+3/2
+1	+1/2	1

		3/2	1/2
		+1/2	+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

		3/2	1/2
		−1/2	−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

		3/2
		−3/2
−1	−1/2	1

3/2 × 1/2

		2
		+2
+3/2	+1/2	1

		2	1
		+1	+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

		2	1
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

		2	1
		−1	−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

		2
		−2
−3/2	−1/2	1

2 × 1

		3
		+3
+2	+1	1

		3	2
		+2	+2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

		3	2	1
		+1	+1	+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

		3	2	1
		0	0	0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

		3	2	1
		−1	−1	−1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

		3	2
		−2	−2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

		3
		−3
−2	−1	1

3/2 × 1

		5/2
		+5/2
+3/2	+1	1

		5/2	3/2
		+3/2	+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

		5/2	3/2	1/2
		+1/2	+1/2	+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

		5/2	3/2	1/2
		−1/2	−1/2	−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

		5/2	3/2
		−3/2	−3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

		5/2
		−5/2
−3/2	−1	1

1 × 1

		2
		+2
+1	+1	1

		2	1
		+1	+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

		2	1	0
		0	0	0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

		2	1
		−1	−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

		2
		−2
−1	−1	1